# Chapter 3 - Connectivity - 2-Connected Graphs

*Recall:* A graph $G$ is called $k$-**connected** (for $k \in \mathbb{N}$) if $|G| > k$ and $G - X$ is connected for every set $X \subseteq V$ with $|X| < k$. Largest $k$ such that $G$ is $k$-connected is called **connectivity** of $G$, denoted $\kappa(G)$.

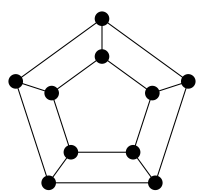
Notice $\kappa(K_n) = n - 1$.

**Goals:**
Describe structure and/or construction of 2- and 3-connected graphs.
Show that in a $k$-connected graph every 2 vertices are connected by $k$ internally disjoint paths.

Let $G$ be a graph and $H_1$ and $H_2$ be a subgraphs of $G$. An $H_1$-$H_2$ **path** is a path with one endpoint in $H_1$ and the other in $H_2$ but edge disjoint with $H_1$ and $H_2$. If $H_1 = H_2$, we call it $H_1$-**path**

**Proposition (Ear decomposition)** A graph is 2-connected if and only if it can be constructed from a cycle by successively adding $H$-paths to graphs $H$ already constructed.

**1:** Show how the $C_5$-prism graph can be constructed in this way.



**2:** Show that every graph constructed this way is 2-connected.

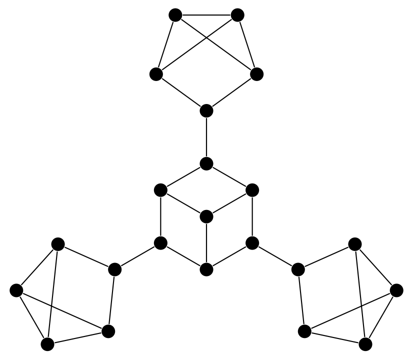
**Solution:** Induction on the number of ears. Cycle is 2-connected. If we add an ear to a 2-connected graph, it is still 2-connected. One needs to check cases of removing one vertex.

**3:** Show the every 2-connected graph can be constructed as in the proposition.
If $G$ is 2-connected, it has a cycle (why?). Take the largest subgraph $H$ of $G$ that can be created by the construction. Show $H$ is induced. How to find a new $H$-path?

**Solution:** If $H$ is not induced then any edge $e$ that is missing to $H$ being induced is and $H$-path. So $H$ is induced.
Next if $H$ is an induced proper subgraph, there exists $uv \in E(G)$ such that $u \in V(H)$ and $v \notin V(H)$. Since $G$ is 2-connected, there is a $v$-$H$ path $P$. Notice that $uP$ is an $H$-path contradicting the maximality of $H$.

A **block** in $G$ is a maximal connected subgraph without a cutvertex. Hence block is either 2-connected or a bridge.

**4:** Identify blocks in the following graph.



**5:** Show that if $B_1$ and $B_2$ are blocks of $G$, then $|V(B_1) \cap V(B_2)| \leq 1$.

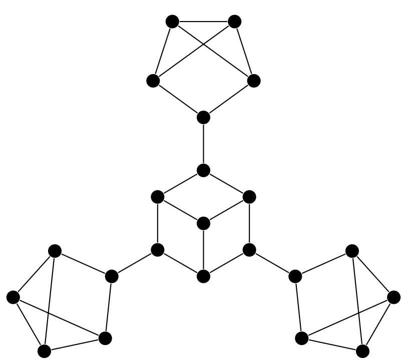
**Solution:** Suppose for contradiction $u, v \in V(B_1) \cap V(B_2)$. Let $H = G[V(B_1) \cup V(B_2)]$. Since $B_i - x$ is connected for all $x \in V(B_i)$, there exists a path from any vertex of $H$ to $y \in \{u, v\} \setminus \{x\}$ in $H - x$. So $H - x$ is connected. Therefore, $H$ is a block. This is a contradiction to $B_1$ and $B_2$ being blocks.

**6:** Show that if $B_1$ and $B_2$ are blocks of $G$ and $v \in V(B_1) \cap V(B_2)$, then $v$ is a cut vertex in $G$.

**Solution:** If $G - v$ has $V(B_1)$ and $V(B_2)$ in the same component, then we would get a contradiction with maximality by taking $V(B_1)$, $V(B_2)$, and $B_1 - B_2$ path in $G - v$.

The **block graph** of G is the bipartite graph with the set of cut vertices of $G$ and the set of blocks of $G$ as its two parts, and $vB$ is an edge if the cut vertex $v$ is in block $B$.

**7:** Create a block graph of this graph.



**8:** Show that the block graph of a connected graph is a tree.

**Solution:** If it contained a cycle, it would contradict the maximality of blocks.

Two paths are **internally disjoint** if they do not have any vertices of degree 2 in common.

**Theorem (Whitney 1932)** A graph $G$ on $n \geq 3$ vertices is 2-connected if and only if for each pair $u, v \in V(G)$ there exist two internally disjoint $u - v$ paths.

**9:** Prove the theorem by induction on distance of $u$ and $v$. What if $u$ and $v$ are adjacent? If $u$ and $v$ are not adjacent, consider $w$, which is the neighbor of $v$ on a shortest $u$-$v$ path and use induction on $u, w$ instead.

**Solution:** If $uv \in E(G)$, there is $u$-$v$ path $P$ in $G - uv$. Now $P$ and $uv$ form the two internally disjoint paths.

For $w$ case, there are internally disjoint $u$-$w$ paths $P_1$ and $P_2$. If $v \in P_1$, then we take subpath $u$-$v$ of $P_1$ and add edge $vw$ to $P_2$ and we are have the two paths we were looking for.

Now we are in the case that neither $P_1$ nor $P_2$ contain $v$. Consider a $u$-$v$ path $P$ in $G - w$. Let $x$ be the vertex on $(P_1 \cup P_2) \cap P$ such that there is no $x$-$v$ part of $P$ does not contain other vertices in $P_1 \cup P_2$. Why $x$ exists? By symmetry, $x \in P_1$. We find the two desired paths as $P_2 + wv$ and a path $u$-$x$ following $P_1$ concatenated with $x$-$v$ path following $P$. Here figure is really needed.

**Lemma (Expansion Lemma)** Let $G$ be a graph and $G'$ be the graph obtained from $G$ by adding a new vertex $v$ with at least $k$ neighbors in $G$. If $G$ is $k$-connected, then $G'$ is $k$-connected.

**10:** Prove the expansion lemma. Consider a separating set $S$ in $G'$ and how it interacts with $v$ and $N(v)$.

**Solution:** Let $S$ be a minimum separating set of $G$. If $v \in S$, then $S = \{v\}$ is a separating set of $G$. This implies $|S| \geq k + 1$. If $v \notin S$ and $N(v) \subseteq S$, then $|S| \geq k$. If v and $N(v)$ are in the same component of $G' - S$, then $S$ is a separating set of $G$. So $|S| \geq k$. Therefore, $G'$ is $k$-connected.

**Theorem** If $G$ is a graph on $n \geq 3$ vertices the following are equivalent:

a) $G$ is 2-connected.

b) For all $u, v \in V(G)$, there are two internally disjoint $u$-$v$ paths.

c) For all $u, v \in V(G)$, there is a cycle containing $u$ and $v$.

d) Every pair of edges in $G$ are in a common cycle and $\delta(G) \geq 2$.

**Proof** Its already been shown that a) $\Leftrightarrow$ b). It is obvious that b) $\Leftrightarrow$ c).

**11:** Assume d). Show c)

**Solution:** Let $u, v \in V(G)$. Since, $\delta(G) \geq 2$, there exist vertices $x, y \in V(G) \setminus \{u, v\}$ such that $ux$ and $vy$ are edges. By d) exists a cycle with $ux$ and $vy$, which is a cycle containing both $u$ and $v$.

**12:** Assume a) and show d). Use the expansion lemma to find a cycle through two given edges.

**Solution:** Since $G$ is 2-connected, $\delta(G) \geq 2$. Let $uv, xy \in E(G)$. Create the graph $G'$ by adding new vertices $w$ and $z$ such that $N(w) = \{u, v\}$ and $N(z) = \{x, y\}$. By the Expansion Lemma, $G'$ is 2-connected also. Therefore, there is a cycle containing $w$ and $z$. By replacing $u, w, v$ and $x, z, y$ on the cycle with $u, v$ and $x, y$, respectively, a cycle for $G$ is created.